**Free EB of Point Charge**

**EM field of moving point charge**

Consider the special case of a point charge (q) who’s position varies with time according to **r**′(t′) = **r**0(t′). Let’s determine what the EM field of the charge would be at a point **r** away, at time t. First, the charge and current densities would be:



Normally we would plug ρ and **j** into Jefimenko’s equations above, but it turns out its actually more convenient to start with the Green’s function expression of φ, and **A**. So starting with φ(**r**,t):



Now to do the integral over time, we will use the following property of derivatives



and consequent property of the delta function:



So…



where we define,



Similarly, we can calculate **A**(**r**,t) for the moving charge:



Now doing the time integral we get, like before,



These formulas are called the Lienard-Wiechert Potentials.



So we can see that there is a modification of φ, and **A**, beyond just that due to taking into account the retarded times. The velocity also has a role to play. Continuing on to the fields themselves…we could differentiate the potentials, but its actually easier to start a little further back…remember that  and so depends on t′. And therefore remember that integrals over t′ in the δ function will involve the 1/f′(f-1(t)) = 1/κr factor. So…



Now let’s integrate by parts on the delta function integral…to do so we have to shift the derivative from the argument of the delta function (equivalent to t for differentiation purposes) to t′. Also remember that R depends on t′ so this will bring down a factor of κ when integrating the δ over t′. So doing this…



Then we have so far,



and now we need to differentiate the left term w/r to t′. So first,



and,



and,



and so,



and,



So then, filling these results into our expression…



After a few pages (should put μ0 in terms of ε0 and put everything over the common denominator κr3, as well as group the r2 and r terms together) I’ve checked that this does simplify to:



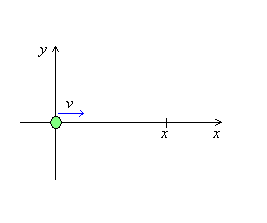
Performing the same procedure for **B** it will become apparent that . So we have,



The term on the left is called the velocity field, and the term on the right is called the acceleration field. Note the acceleration field goes as 1/r, and is descriptive of radiation. If we have more complicated charge distributions, we could just integrate these equations over the charge distribution to obtain the EM field. Probably wouldn’t want to though.

**Example: Electric field of particle moving along x-axis**

What is the electric and magnetic potential as a function of time, along the x-axis, of a charge moving along the x-axis with speed v? Let t = 0 be when the particle crosses the origin.





The retarded time is time at which the particle emitted the light ray that hit the point x at time t. This is given by,



The retarded position is the position of the particle at the retarded time. This is:



Continuing…



and so filling all these in…



The vector potential would be:



Forming the electric field then, we have:



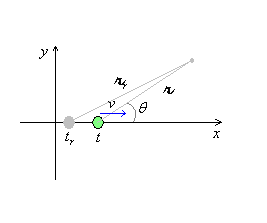
Now let’s work it out explicitly using the formula above. We get,



which is the same. And these results are what we’d expect since when β → 0, the formula reduces to the static result, upon recognizing x′ = vt. Trivially rather, since we see that the field along the axis is along the axis, there is no magnetic field along the axis since **r×E** = 0.

**Example: Electric field, magnetic field at points off center**

Let’s work out the EM field at all points for a charge moving along the x-axis with speed v.



First we have to work out the retarded time and position. The position as a function of time is: . The retarded time is the time when the light would hit the point. Let’s make some manipulations first,



Using the quadratic formula…



OK, well continuing on let’s try to evaluate . This is:



Looking ahead we’ll need,



And we’ll also need:



Finally, we need,



And now we can evaluate the potential. It is:



and the magnetic potential is:



And now let’s determine the electric and magnetic fields. We could take the gradient, etc., but I guess we’ll try the formulas for the fields:



Need to figure out the numerator vector,



and in the denominator we have,



So putting it together we have,



And the magnetic field would be given by:



So for a charge, q, moving at constant velocity, **v**, the EB field is:



Curiously enough, even though the field depends on where the charge was at the retarded time – it still point radially away for **E** and curls around **v** for **B**, just like before. But it isn’t radially symmetric. Looks like it’s largest for points perpendicular to the velocity, smallest for points more aligned with the velocity. As v0 → 0, the result reduces to the usual static result. Also note that along the direction of **v**, the formula is the same as the static result, as we saw above.

**Power radiated by accelerating charge**

Let’s consider the energy radiated by an accelerating charge. A common example is quickly moving charged particles being deccelerated by collisions with other particles. The field for an accelerating charge is:



The power radiated is:



(this d**S** is surface area) When we form the integrand, note that all terms will drop out except the ones which go as 1/r. This is how we can identify these terms as the ones that describe the radiation. Let’s form the expression S = **E**×**B**/μ0 and see what we get,



Now we want to calculate the total power leaving the charge as a function of time. This isn’t quite S above, which is the total power entering a surface area per unit time. So we’ll change variables to tr, and get,



So κS(tr) is the power that is leaving the charge. So altogether we have now,



and then the radiated power distribution would be , using,



where dΩ is the solid angle,



**Example: Power radiated from accelerating charge**

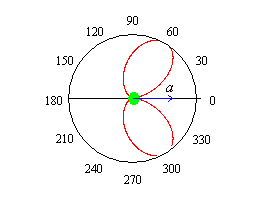
What is the radiative power distribution for a charge moving along a straight line – z axis – with velocity **v** and acceleration **a**?



So we have,



The radiation pattern looks like the illustration below. Note that it is peaked at some angle in the forward direction, but 0 precisely along the direction of motion.



If the v = 0, then the radiation will be peaked at the angle θ = 90˚. As v increases the radiation peaks at an angle closer and closer to 0. Let’s next find the total power radiated. We have to integrate over the all angles,



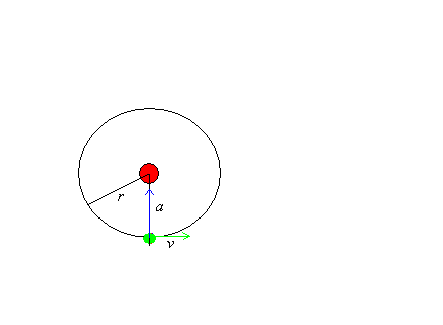
So we have:



We can see that the power radiated goes to ∞ as the velocity of the particle approaches that of light. In the other extreme, where the velocity is much less than that of light (β0 ≈ 0), we get Larmour’s formula.



What if the charge is accelerating around a circle?



Let’s orient our coordinate system at the particle and with the z-axis pointing along **a** as before



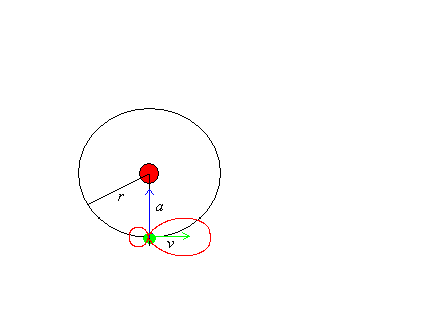
We can get the angular distribution by performing all the cross products. The distribution is similar to that of the linear particle, keeping in mind that the distribution is oriented about **a**. We will get,



Once we get that, we can integrate over all angles to get the total power leaving,



In the small β0 limit, the total power radiated would be according to Larmour’s formula still.



As you can see, most of the radiation is in the direction of the particle’s velocity. This radiation carries with it momentum and thus the particle will experience a significant back-reaction force, which we’ll talk about later.

**Example: Dipole radiation**

Consider a charge oscillating linearly back and forth along a rod, with position given by:

z(t) = dcos(ωt). What is the total power radiated?

We can use the linearly accelerating charge formula as a basis for this calculation. The acceleration of the charge would be given by a(t) = -dω2cos(ωt). Plugging this into the formula, the instantaneous rate of power radiation would be:



The average of cos2(ωt) over a period is ½, so the average power radiated would be:



where p is the dipole moment.

**Radiation Reaction Force**

The Larmour formula gives us the energy leaving an accelerating particle. Energy flux in a particle direction is equivalent to a backwards reaction force, just like how a rocket pushing matter out of its end propels it in the opposite direction. This force we can try to characterize as follows. We can demand first that the work done on the particle by the reaction force is the same as the energy leaving the particle in the first place, between any two times t1 and t2. So we must have,



(ignoring the boundaries). Thus we may equate



The formula should be taken with a grain of salt however. For one, physical laws can only be a function of the 2nd derivative of **r** – according to Ostragradsky or something. For instance, runaway solutions that blow are an unavoidable consequence of writing it like this. It is probably illegitimate to make this force identification, and probably only applies underneath the integral sign, which would curb its pathological tendencies. Nonetheless, it can be used to estimate corrections to the classical behavior.